On the discontinuous second–order deviated Dirichlet problem with non–monotone conditions

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January 21, 2013

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Abstract

We provide a new result on the existence of extremal solutions for second–order Dirichlet problems with deviation argument. As a novelty in this work, the nonlinearity need not be continuous or monotone. In order to obtain this new result, we use a generalized monotone method coupled with lower and upper solutions.

Primary classification number: 34B99.

1 Introduction

Differential equations with deviated arguments have received a lot of attention in last years. Because of their meaningful interest for modelling real–life processes, see for instance [4], [5], [8], [13] and references therein, nowadays mathematicians work a lot in order to study both qualitative and quantitative properties of these equations.

In the present work, we look for a new result on the existence and location of extremal solutions for the following second–order Dirichlet problem with deviated argument:

$$\begin{cases} -u''(t) = f(t, u(t), u(\tau(t))), \ t \in I = [0, T]; \\ u(t) = \phi(t), \ t \in [-r, 0], \quad u(T) = B, \end{cases} \tag{1.1}$$

where ϕ is a continuous start function and τ is measurable and such that $\tau(I) \subset [-r, T]$. So this framework includes, in particular, delayed equations. We will search for such solutions inside the set

$$\mathcal{X} = \{u \in \mathcal{C}([-r, T]) : u_{|I} \in W^{2,1}(I)\}.$$

Problem (1.1) has been studied recently from different points of view: in [7] the existence of solutions for a more general problem was obtained via application of Schauder's Theorem, assuming so that nonlinearity f is continuous with respect to all variables. In [6] continuity was replaced by monotone conditions, and this change allowed us to guarantee the existence of extremal solutions between lower and upper solutions. On the other hand, in [11] the author studied (1.1) with periodic conditions and with the assumption

$\tau(I) \subset I$. His approach combines continuiuty and one–sided Lipschitz conditions in order to avoid monotonicity, by using a similar approach to that done in [14] for a first–order problem.

The main goal of the present paper follows the line of [11], but we improve those results in the following ways: we require no continuity conditions, we let deviated argument τ to take values outside I and one–side Lipschitz constants are replaced by L^p- functions. All these contributions allow us to study a larger class of problems, including classical delay problems, as we will show in last section with some examples. In order to do that, we use a generalized monotone method coupled with lower and upper solutions.

This paper is organized as follows: In Section 2 we provide a result on the existence of a unique solution for problem (1.1) in the case that nonlinearity is Lipschitz–continuous with respect to spatial variables. This result is used later, in Section 3, for solving auxiliar linear approximations of problem (1.1). Section 2 is devoted to a comparison result for second–order Dirichlet problems with deviation. In Section 3 we include the main result on this work. There, we use the work from previous sections in order to define an adequate fixed–point operator in a certain functional interval. This operator will provide us the extremal solutions for (1.1). Finally, in Section 4 we include some examples of application of our results.

In the sequel, $\tau : I \longrightarrow [-r, T]$ is a measurable deviated argument.

2 Uniqueness result

As an auxiliar step in the process of linearization of problem (1.1), we provide now a result on the existence of unique solutions for problem (1.1) in the case that nonlinearity f is Lipschitz–continuous.

THEOREM 2.1 *Assume that function $f : I \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ satisfies the following conditions:*

(H_1) *For each $x, y \in \mathbb{R}$, the mapping $t \in I \longmapsto f(t, x, y)$ is measurable;*

(H_2) *For each compact subset $K \subset \mathbb{R}^2$ there exists $\psi_K \in L^1(I, [0, +\infty))$ such that*

$$|f(t, x, y)| \le \psi_K(t) \quad \textit{for all } (x, y) \in K;$$

(H_3) *There exist nonnegative functions L_1, L_2 such that*

$$|f(t, x, y) - f(t, \overline{x}, \overline{y})| \le L_1(t)|x - \overline{x}| + L_2(t)|y - \overline{y}|.$$

Moreover, functions L_1, L_2 satisfy one of the following:

(C_1) $L_1, L_2 \in L^\infty(I)$ *and* $||L_1 + L_2||_\infty < \dfrac{1}{T^2}$;

(C_2) $L_1, L_2 \in L^2(I)$ *and* $||L_1 + L_2||_2 < \left(\dfrac{3}{2T^3}\right)^{1/2}$;

(C_3) $L_1, L_2 \in L^1(I)$ *and* $||L_1 + L_2||_1 < \dfrac{1}{2T}$.

Under these conditions, problem (1.1) has a unique solution in $\mathcal{X}$.

Proof. First of all, notice that $u \in \mathcal{X}$ is a solution of problem (1.1) if and only if u is a fixed point of the operator $G : \mathcal{C}([-r,T]) \longrightarrow \mathcal{C}([-r,T])$ defined as

$$\begin{cases} Gu(t) = \phi(t), \quad t \in [-r,0], \\ Gu(t) = \phi(0) + Ct - \displaystyle\int_0^t (t-s) f(s,u(s),u(\tau(s)))\, ds, \quad t \in [0,T], \end{cases}$$

where

$$C = \frac{1}{T}\left(B - \phi(0) + \int_0^T (T-s) f(s,u(s),u(\tau(s)))\, ds\right).$$

Notice that operator G is well–defined by virtue of conditions (H_1) and (H_2). So, we will show now that G is a contraction when considering the Banach space $\mathcal{C}([-r,T])$ equipped with its usual supremum norm, $||u|| = \max_{t\in[-r,T]} |u(t)|$:

Given $u, v \in \mathcal{C}([-r,T])$, for each $t \in I$ it is

$$|Gu(t) - Gv(t)| \le \frac{t}{T}\left(\int_0^T (T-s)|f(s,u(s),u(\tau(s))) - f(s,v(s),v(\tau(s)))|\, ds\right) \tag{2.2}$$

$$+ \int_0^t (t-s)|f(s,u(s),u(\tau(s))) - f(s,v(s),v(\tau(s)))|\, ds, \tag{2.3}$$

and then

$$||Gu - Gv|| \le 2\left(\int_0^T (T-s)(L_1(s) + L_2(s))\, ds\right) ||u - v||.$$

Now notice that:

1. If (C_1) holds, then

$$\int_0^T (T-s)(L_1(s) + L_2(s))\, ds \le ||L_1 + L_2||_\infty \frac{T^2}{2} < \frac{1}{2},$$

so G is a contraction;

2. If (C_2) holds, then

$$\int_0^T (T-s)(L_1(s) + L_2(s))\, ds \le ||L_1 + L_2||_2^2 \frac{T^3}{3} < \frac{1}{2},$$

so G is a contraction;

3. If (C_3) holds, then

$$\int_0^T (T-s)(L_1(s) + L_2(s))\, ds \le T\, ||L_1 + L_2||_1 < \frac{1}{2},$$

so G is a contraction;

By application of Banach's fixed–point theorem, operator G has a unique fixed point which is the unique solution of problem (1.1) in $\mathcal{X}$. □

3 Maximum principle and main result

The first step on developing our generalized monotone method is to obtain a comparison result for second–order Dirichlet problems with delay. The proof of this result follows the line of [12, Theorem 2.2] and [14, Theorem 2.1].

LEMMA 3.1 *Assume that $p \in \mathcal{X}$ and there exist nonnegative functions L_1, L_2 satisfying the following:*

(M_1) $-p''(t) + L_1(t)p(t) + L_2(t)p(\tau(t)) \geq 0$ *for a.a.* $t \in [0, T]$*;*

(M_2) $p(0) \geq 0$, $p(T) \geq 0$*;*

(M_3) $0 \leq p(t) \leq p(0)$ *for all* $t \in [-r, 0]$*;*

(M_4) *functions L_1, L_2 satisfy one of the following conditions:*

- $(\hat{C_1})$ $L_1, L_2 \in L^\infty(I)$ *and* $||L_1 + L_2||_\infty < \dfrac{2}{T^2}$*;*
- $(\hat{C_2})$ $L_1, L_2 \in L^2(I)$ *and* $||L_1 + L_2||_2 < \dfrac{\sqrt{2}}{T}$*;*
- $(\hat{C_3})$ $L_1, L_2 \in L^1(I)$ *and* $||L_1 + L_2||_1 < \dfrac{1}{T}$

Then $p(t) \geq 0$ for all $t \in [-r, T]$.

Proof. Assume that there exists t_0 in $(0, T)$ such that $p(t_0) < 0$. We consider two cases:
Case I: $p(t) \leq 0$ for all $t \in [0, T]$ and p is not identically 0 in that interval. In this case, $p(0) = p(T) = 0 = p(t)$ for $t \in [-r, 0]$, and condition (M_1) provides that

$$p''(t) \leq L_1(t)p(t) + L_2(t)p(\tau(t)) \leq 0 \quad \text{for a.a. } t \in (0, T),$$

so $p \equiv 0$, a contradiction.
Case II: there exist $t_1, t_2 \in (0, T)$ such that $p(t_1) > 0$ and $p(t_2) < 0$. In this case, there exists $t_3 \in (0, T)$, $t_4 \in [0, T]$ satisfying

$$p(t_3) = \min_{t \in [-r, T]} p(t) < 0, \quad p'(t_3) = 0,$$

$$p(t_4) = \max_{t \in [-r, T]} p(t) > 0.$$

So,

$$p''(t) \leq (L_1(t) + L_2(t))p(t_4), \quad \text{for a.a. } t \in [0, T]. \tag{3.4}$$

If $t_4 < t_3$ then we take $s \in (t_4, t_3)$ and integrate (3.4) from s to t_3 to obtain

$$-p'(s) \leq p(t_4) \int_s^{t_3} (L_1(t) + L_2(t))\, dt,$$

and now integrating from t_4 to t_3:

$$p(t_4) \leq p(t_4) - p(t_3) \leq p(t_4) \int_{t_4}^{t_3} \int_s^{t_3} (L_1(t) + L_2(t))\, dt\, ds,$$

and then

$$1 \le \int_{t_4}^{t_3} \int_s^{t_3} (L_1(t) + L_2(t))\, dt\, ds. \tag{3.5}$$

Now notice that we can bound integral in (3.5) by the following numbers:
If condition $(\hat{C_1})$ holds, then

$$\int_{t_4}^{t_3} \int_s^{t_3} (L_1(t) + L_2(t))\, dt\, ds \le \frac{T^2}{2}\, ||L_1 + L_2||_\infty < 1,$$

which is a contradiction.
If condition $(\hat{C_2})$ holds, then

$$\int_{t_4}^{t_3} \int_s^{t_3} (L_1(t) + L_2(t))\, dt\, ds \le \frac{T^2}{2}\, ||L_1 + L_2||_2^2 < 1,$$

again a contradiction.
If condition $(\hat{C_3})$ holds, then

$$\int_{t_4}^{t_3} \int_s^{t_3} (L_1(t) + L_2(t))\, dt\, ds \le T\, ||L_1 + L_2||_1 < 1.$$

On the other hand, if $t_3 < t_4$, then we reason like above, now integrating from t_3 to s $(t_3 < s < t_4)$ and then from t_3 to t_4 to get the same contradictions. □

Remark 3.1 *Notice that for* $\frac{3}{4} \le T$ *each condition* (C_i) *implies* $(\hat{C_i})$, $i = 1, 2, 3$, *and for* $0 < T \le \frac{3}{4}$ *we have* $(C_1) \Rightarrow (\hat{C_1})$, $(C_3) \Rightarrow (\hat{C_3})$ *and* $(\hat{C_2}) \Rightarrow (C_2)$.

Now we introduce a Lemma on extremal fixed–points for monotone operators in ordered sets. This result will play an essential role in our future argumentations.

Lemma 3.2 *[10, Theorem 1.2.2] Let* Y *be a subset of an ordered metric space* X, $[a, b]$ *a nonempty interval in* Y *and* $G : [a, b] \longrightarrow [a, b]$ *a nondecreasing operator. If* $\{Gx_n\}_{n=1}^\infty$ *converges in* Y *whenever* $\{x_n\}_{n=1}^\infty$ *is a monotone sequence in* $[a, b]$, *then operator* G *has the greatest,* x^*, *and the least,* x_*, *fixed point in* $[a, b]$. *Moreover, we have that*

$$x_* = \min\{x : Gx \le x\}, \quad x^* = \max\{x : x \le Gx\}.$$

We define now what we mean by lower and upper solutions for problem (1.1).

Definition 3.1 *We say that* $\alpha, \beta \in \mathcal{X}$ *are, respectively, a lower and an upper solution for problem (1.1) if the compositions*

$$t \in [0, T] \longmapsto f(t, \alpha(t), \alpha(\tau(t))), \quad t \in [0, T] \longmapsto f(t, \beta(t), \beta(\tau(t)))$$

are measurable and the following conditions hold:

$$\begin{cases} -\alpha''(t) \le f(t, \alpha(t), \alpha(\tau(t))) \text{ for a.a. } t \in [0, T], \\ \alpha(t) \le \phi(t),\ \phi(t) - \alpha(t) \le \phi(0) - \alpha(0) \text{ for all } t \in [-r, 0], \quad \alpha(T) \le B; \end{cases}$$

$$\begin{cases} -\beta''(t) \ge f(t, \beta(t), \beta(\tau(t))) \text{ for a.a. } t \in [0, T], \\ \beta(t) \ge \phi(t),\ \beta(t) - \phi(t) \le \beta(0) - \phi(0) \text{ for all } t \in [-r, 0], \quad \beta(T) \ge B. \end{cases}$$

Theorem 3.1 *Assume that there exist $\alpha, \beta \in \mathcal{X}$ which are, respectively, a lower and an upper solution for problem (1.1), with $\alpha(t) \le \beta(t)$ for all $t \in [-r, T]$, and put*

$$[\alpha, \beta] = \{\gamma \in \mathcal{C}([-r, T]) \,:\, \alpha(t) \le \gamma(t) \le \beta(t) \ \ \textit{for all } t \in [-r, T]\}.$$

Assume, moreover, that the following conditions hold:

(H_1) *For each $\gamma \in [\alpha, \beta]$ the composition $t \in I \longmapsto f(t, \gamma(t), \gamma(\tau(t)))$ is measurable;*

(H_2) *There exists $\psi \in L^1(I, [0, \infty))$ such that for a.a. $t \in I$, all $x \in [\alpha(t), \beta(t)]$ and all $y \in [\alpha(\tau(t)), \beta(\tau(t))]$ we have $|f(t, x, y)| \le \psi(t)$;*

(H_3) *There exist nonnegative functions L_1, L_2 such that for a.a. $t \in I$*

$$f(t, \overline{x}, \overline{y}) - f(t, x, y) \ge -L_1(t)(\overline{x} - x) - L_2(t)(\overline{y} - y)$$

whenever $\alpha(t) \le x \le \overline{x} \le \beta(t)$, $\alpha(\tau(t)) \le y \le \overline{y} \le \beta(\tau(t))$.

Moreover, if $T \ge \dfrac{3}{4}$ then functions L_1, L_2 satisfy one of the conditions (C_1), (C_2), (C_3) and if $0 < T < \dfrac{3}{4}$ then they satisfy one of the following: (C_1), $(\hat{C_2})$, (C_3).

In these conditions, problem (1.1) has the extremal solutions in $[\alpha, \beta]$.

Proof. We define an operator $G : [\alpha, \beta] \longrightarrow [\alpha, \beta]$ as follows: for each $\gamma \in [\alpha, \beta]$, $G\gamma$ is the unique solution of the problem

$$\begin{cases} -u''(t) + L_1(t)(u(t) - \gamma(t)) + L_2(t)(u(\tau(t)) - \gamma(\tau(t))) = f(t, \gamma(t), \gamma(\tau(t))), & \text{for a.a. } t \in I, \\ u(t) = \phi(t) \ \text{ for all } t \in [-r, 0], \quad u(T) = B. \end{cases} \tag{3.6}$$

Step 1: Operator G is well–defined from $[\alpha, \beta]$ to $\mathcal{X}$. We can rewrite the differential equation in (3.6) in the form $-u''(t) = \widetilde{f}(t, u(t), u(\tau(t)))$, where

$$\widetilde{f}(t, x, y) = -L_1(t)x - L_2(t)y + f(t, \gamma(t), \gamma(\tau(t))) + L_1(t)\gamma(t) + L_2(t)\gamma(\tau(t)),$$

so problem (3.6) satisfies conditions of Theorem 2.1 and then it has a unique solution.

Step 2: Operator G is nondecreasing. Let $\gamma_1, \gamma_2 \in [\alpha, \beta]$ such that $\gamma_1(t) \le \gamma_2(t)$ for all $t \in [-r, T]$. We will show now that $G\gamma_1(t) \le G\gamma_2(t)$ for all t.
First of all, notice that $G\gamma_1, G\gamma_2$ satisfy for a.a. $t \in I$ that

$$-G\gamma_1''(t) + L_1(t)(G\gamma_1(t) - \gamma_1(t)) + L_2(t)(G\gamma_1(\tau(t)) - \gamma_1(\tau(t)))) = f(t, \gamma_1(t), \gamma_1(\tau(t))),$$

$$-G\gamma_2''(t) + L_1(t)(G\gamma_2(t) - \gamma_2(t)) + L_2(t)(G\gamma_2(\tau(t)) - \gamma_2(\tau(t)))) = f(t, \gamma_2(t), \gamma_2(\tau(t))),$$

and, moreover,

$$G\gamma_1(t) = G\gamma_2(t) = \phi(t) \ \text{ for all } t \in [-r, 0], \quad G\gamma_1(T) = G\gamma_2(T) = B.$$

So, by virtue of condition (H_3) we obtain that function $p = G\gamma_2 - G\gamma_1$ satisfies:

$$\begin{cases} -p''(t) + L_1(t)p(t) + L_2(t)p(\tau(t)) \ge 0, & \text{for a.a. } t \in I, \\ p(0) = p(T) = 0 = p(t) \ \text{ for all } t \in [-r, 0]. \end{cases}$$

Then, by application of Lemma 3.1 we obtain that $p \geq 0$ on $[-r, T]$, so $G\gamma_1(t) \leq G\gamma_2(t)$ for all $t \in [-r, T]$, which implies that operator G is nondecreasing.

Step 3: $G([\alpha, \beta]) \subset [\alpha, \beta]$. Because of being α a lower solution for problem (1.1), we have that $p = G\alpha - \alpha$ satisfies:

$$\begin{cases} -p''(t) + L_1(t)p(t) + L_2(t)p(\tau(t)) \geq 0 \quad \text{for a.a. } t \in I, \\ p(t) = \phi(t) - \alpha(t) \geq 0, \quad p(t) = \phi(t) - \alpha(t) \leq \phi(0) - \alpha(0) = p(0) \quad \text{for all } t \in [-r, 0], \\ p(T) = \phi(T) - \alpha(t) = T - \alpha(t) \geq 0. \end{cases}$$

So, by application of Lemma (3.1) we conclude that $G\alpha(t) \geq \alpha(t)$ for all $t \in [-r, T]$. In the same way we prove that $G\beta \leq \beta$. This and monotonicity of operator G imply that $G([\alpha, \beta]) \subset [\alpha, \beta]$.

Step 4: $\{G\gamma_n\}_{n=1}^{\infty}$ converges in $\mathcal{C}([-r, T])$ whenever $\{\gamma_n\}_{n=1}^{\infty}$ is a monotone sequence in $[\alpha, \beta]$. Put $z_n = G\gamma_n \in \mathcal{C}([-r, T])$. Because of the monotonicities of sequence $\{\gamma_n\}_{n=1}^{\infty}$ and operator G we obtain that $\{z_n\}_{n=1}^{\infty}$ is a monotone sequence in $[\alpha, \beta]$, so it has it pointwise limit, say z.
As $\{z_n(t)\}_{n=1}^{\infty}$ is a constant sequence for $t \in [-r, 0]$, $\{z_n\}_{n=1}^{\infty}$ converges to z uniformly in that interval. Now, fixed $t \in I$, by virtue of Mean Value Theorem there exists $c = c(n) \in [0, T]$ such that

$$z_n'(t) = \frac{B - \phi(0)}{T} + \int_c^t \widetilde{f}(s, z_n(s), z_n(\tau(s)))\, ds,$$

so

$$||z_n'||_\infty \leq \frac{|B - \phi(0)|}{T} + ||\psi||_1 + ||L_1 + L_2||_1 ||\beta - \alpha||_\infty.$$

Then, the sequence $\{z_n'\}_{n=1}^{\infty}$ is uniformly bounded on I, so $\{z_n\}_{n=1}^{\infty}$ converges to z in $\mathcal{C}(I)$.

Step 5: Problem (1.1) has the extremal solutions in $[\alpha, \beta]$. By application of Lemma 3.1, operator G has in $[\alpha, \beta]$ the extremal fixed points, u^*, u_*. We will show now that u^*, u_* correspond, respectively, with the greatest and the least solution of problem (1.1) in $[\alpha, \beta]$. First of all, it is clear that each fixed point of G is also a solution of (1.1). On the other hand, if u is a solution of (1.1) between α and β then u also solves (3.6), and the uniqueness of solution of problem (3.6) provides that $Gu = u$. Then, $u_* \leq u \leq u^*$, so u_*, u^* are the extremal solutions of (1.1) in $[\alpha, \beta]$. □

Remark 3.2 *Among all conditions in the previous result, perhaps (H_1) is the most difficult to check in practise when working with discontinuous nonlinearities. The reader is referred to [1], [3], [9] for a larger disquisition about this. In those references, some results for guaranteing (H_1) are provided. On the other hand, and roughly speaking, condition (H_3) forces f not to have downwards discontinuities. In Section 4 we introduce a Lemma which will be useful on checking this condition in examples.*

4 Examples of application

In this section we show some examples of application of Theorem 3.1. As far as we know, the following problems can be studied with no result in the literature. As a technical support in order to check condition (H_3), we begin by introducing a simple lemma.

LEMMA 4.1 *Let $\{x_k\}_{k=1}^{\infty}$ a strictly increasing sequence of real numbers and assume that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is such that $f_{|(x_{k-1},x_k)} \in \mathcal{C}^1(x_{k-1}, x_k)$ for all $k = 1, 2, \ldots$ Assume moreover that the following conditions hold for each $k \in \mathbb{N}$:*

(*i*)

$$\lim_{x \to x_k^-} f(x) \leq f(x_k) \leq \lim_{x \to x_k^+} f(x);$$

(*ii*) *there exist $M_k = \inf\{f'(x) \,:\, x \in (x_{k-1}, x_k)\}$ and $M = \inf\{M_k \,:\, k \in \mathbb{N}\}$.*

In these conditions, the function $g : x \in \mathbb{R} \longmapsto g(x) = f(x) + |M|x$ is nondecreasing.

Proof. Fixed $k \in \mathbb{N}$ we have that g is differentiable in (x_{k-1}, x_k) and $g'(x) \geq M_k + |M| \geq 0$ for all $x \in (x_{k-1}, x_k)$, so g is nondecreasing in that interval. On the other hand, condition (i) provides that f is nondecreasing at point x_k, and so g. $\square$

EXAMPLE 4.1 *Consider the following problem with delay ($[\cdot]$ means integer part):*

$$\begin{cases} -u''(t) = [t\,u(t)] - \dfrac{1}{9}u(t-1)\,\sin\left(\dfrac{u(t-1)\pi}{2[|u(t-1)|]+2}\right) \equiv f(t, u(t), u(t-1)) \quad a.e. \text{ on } I = [0,2], \\ \\ u(t) = \phi(t) = \cos\dfrac{\pi}{2}\,t, \;\; t \in [-1, 0], \quad u(2) = \dfrac{\pi}{4}. \end{cases} \tag{4.7}$$

Notice that function f is discontinuous with respect to its three variables and it is non–monotone with respect to its functional one. We will show that problem (4.7) has extremal solutions between adequate lower and upper solutions.

Consider the functions

$$\alpha(t) = 0 \quad \text{for all } t \in [-1, 2], \tag{4.8}$$

$$\beta(t) = \begin{cases} \cos\dfrac{\pi}{2}\,t, & \text{if } t \in [-1, 0], \\ \\ 1 - t(t-2), & \text{if } t \in [0, 2], \end{cases} \tag{4.9}$$

We will prove that α and β are, respectively, a lower and an upper solution for problem (4.7). First of all, notice that $\alpha(t) \leq \phi(t) \leq \beta(t)$ for all $t \in [-1, 0]$ and that the compositions $t \in [0, 2] \longmapsto f(t, \alpha(t), \alpha(t-1))$ and $t \in [0, 2] \longmapsto f(t, \beta(t), \beta(t-1))$ have at most a null set of discontinuity points, so they are both measurable. Now, taking into account for $t \in [0, 2]$ it is $t - t^2(t-2) < 3$, we have for a.a. $t \in [0, 2]$ that

$$f(t, \alpha(t), \alpha(t-1)) = f(t, 0, 0) = 0 = -\alpha''(t),$$

$$f(t, \beta(t), \beta(t-1)) = [t - t^2(t-2)] - \frac{1}{9}\beta(t-1)\sin\left(\frac{\beta(t-1)\pi}{2[|\beta(t-1)|]+2}\right) \leq 2 = -\beta''(t).$$

Then, α and β are lower and upper solutions for our problem, which moreover satisfy $\alpha \leq \beta$ on $[-1, 2]$.

Reasoning as above, if $\gamma \in [\alpha, \beta]$, with $[\alpha, \beta]$ defined as in Theorem 3.1, then the composition $t \in [0, 2] \longmapsto f(t, \gamma(t), \gamma(t-1))$ is measurable, so condition (H_1) in that Theorem holds.

On the other hand, for $t \in [-1,2]$ we have that $0 \le \alpha(t) \le \beta(t) \le 3$, so condition (H_2) holds with $\psi \equiv 4$.

We are going now to check (H_3). First, notice that for $t \in [0,2]$ function $f(t,\cdot,y) = [tx]$ is nondecreasing, so we can take $L_1 \equiv 0$. On the other hand, for $t \in [0,2]$ and $x \in [\alpha(t), \beta(t)]$ we have that $f(t,x,\cdot)$ is discontinuous exactly at integer numbers, with

$$\lim_{y \to k^-} f(t,x,y) = -\frac{1}{9} y \sin\left(\frac{\pi}{2}\right)$$

and

$$f(t,x,k) = \lim_{y \to k^+} f(t,x,y) = -\frac{1}{9} y \sin\left(\frac{k}{k+1}\,\frac{\pi}{2}\right),$$

for $k \in \mathbb{Z}$. So, condition (i) in Lemma 4.1 is satisfied for $y \ge 0$ (in particular, inside the interval $[\alpha(t-1), \beta(t-1)]$). Moreover, for $y \in (k-1,k)$, $k \ge 1$, it is

$$\frac{\partial}{\partial y} f(t,x,y) = \frac{-1}{9}\left(1 + \frac{y\pi}{2k}\right)\cos\left(\frac{y\pi}{2k}\right) \ge -\frac{1}{9}\left(1 + \frac{\pi}{2}\right),$$

and by application of Lemma 4.1 condition (H_3) is satisfied with $L_2 \equiv \frac{1}{9}\left(1 + \frac{\pi}{2}\right)$. Notice that $\frac{1}{9}\left(1 + \frac{\pi}{2}\right) \approx 0.2857$, so nor condition (C_1) nor (C_3) are satisfied. However, $\sqrt{2}\, L_2 < \frac{\sqrt{3}}{4}$, so condition (C_2) holds.

As all the conditions in Theorem 3.1 hold, we conclude that problem (4.7) has the extremal solutions between α and β.

Example 4.2 *Fix $0 < k < \frac{1}{10}$, let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ the function defined by*

$$\varphi(0) = 0, \quad \varphi(x) = \frac{k}{n} - kx \quad \text{if } |x| \in \left(\frac{1}{n+1}, \frac{1}{n}\right] \bigcup (n, n+1],\ n = 1, 2, \ldots,$$

and consider the following problem with both delay and advance:

$$\begin{cases} -u''(t) = \sin(t) + \varphi(u(\sqrt{1-t})) + \dfrac{1}{5\sqrt{t}} u(\sqrt{t}) \equiv f(t, u(\sqrt{1-t}), u(\sqrt{t})) & \text{for a.a. } t \in I = [0,1], \\ u(0) = u(1) = 0. \end{cases} \tag{4.10}$$

Notice that in problem (4.10) the nonlinearity f is discontinuous and non–monotone with respect to its second variable in a countable set. Moreover, it blows up when $t \to 0$.

Consider now the functions $\alpha(t) = t^2 - t = -\beta(t)$ for $t \in [0,1]$. We will show that α and β are, respectively, a lower and an upper solution for problem (4.10).

First, notice that if γ is a continuous function then $\varphi \circ \gamma$ is discontinuous at most in a countable set, so $\varphi \circ \gamma$ is measurable. On the other hand, $\alpha(0) = \alpha(1) = 0 = \beta(0) = \beta(1)$, and for a.a. $t \in I$ we have that

$$f(t, \alpha(\sqrt{1-t}), \alpha(\sqrt{t})) = \sin(t) + \varphi(1 - t - \sqrt{1-t}) + \frac{\sqrt{t} - 1}{5} \ge -1 - 1/5 \ge -2 = -\alpha''(t),$$

$$f(t,\beta(\sqrt{1-t}),\beta(\sqrt{t})) = \sin(t)+\varphi(\sqrt{1-t}-(1-t))+\frac{1-\sqrt{t}}{5} \le 1+k/2+1/5 \le 2 = -\beta''(t).$$

Then, α, β are, repectively, a lower and an upper solution for (4.10).

Now notice that $\frac{-1}{4} \le \alpha(t) \le \beta(t) \le \frac{1}{4}$ for all $t \in I$ and for $x, y \in \left[\frac{-1}{4}, \frac{1}{4}\right]$ we have

$$|f(t,x,y)| \le \psi(t) = \sin(t) + \frac{k}{2} + \frac{1}{20\sqrt{t}},$$

with $\psi \in L^1(I)$, so condition (H_2) holds.

Finally, condition (H_3) is satisfied with $L_1(t) = k$ and $L_2(t) = \frac{1}{5\sqrt{t}}$, $t \in (0,1]$, and we have that

$$||L_1 + L_2||_1 = k + \frac{2}{5} < \frac{1}{2} \quad \text{if } k < \frac{1}{10}.$$

By application of Theorem 3.1, problem (4.10) has the extremal solutions in $[\alpha, \beta]$.

REMARK 4.1 *Notice that, in problem 4.10, there exists no constant $\hat{L}$ such that*

$$y \in [\alpha(\sqrt{t}), \beta(\sqrt{t})] \longmapsto f(t,x,y) + \hat{L}y$$

is nondecrasing, so our improvement (from constant one–sided Lipschitz conditions to L^p- ones) become essential in this case.

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